Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 18, 19

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Textbook Sections: 11.2, 11.4, 11.5

Today's Topic: Convergence Tests for Series

Recall from Monday:

- Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$ is called an *infinite series*
- To assign a value to $\sum_{n=1}^{\infty} a_n$, we
  1) Consider the *partial sums* $S_N = \sum_{n=1}^{N} a_n$
  2) Investigate the sequence $\{S_N\}_{N=1}^{\infty}$:
     - If $\lim_{N\to\infty} S_N = S$ exists, then we say
     $$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = S$$
     and that $\sum a_n$ is *convergent*.
     - If $\{S_N\}$ diverges, then we say $\sum a_n$ *diverges*.
- As an example, we introduced *geometric series* $\sum_{n=1}^{\infty} ar^{n-1}$, and showed that in this case $S_N = a\frac{1-r^N}{1-r}$. Thus,
$$\sum_{n=1}^{\infty} ar^{n-1} = \lim_{N\to\infty} S_N = \begin{cases} \frac{a}{1-r} & \text{if } r\in(-1,1) \\ \text{does not exist (divergence)} & \text{otherwise} \end{cases}$$

Today let's think a bit more generally.

When answering whether or not a given series $\sum_{n=1}^{\infty} a_n$ converges, it is very useful to first ask the following question:

Do the terms $a_n \to 0$ as $n \to \infty$?

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ <u>cannot</u> possibly converge!

Proof: If $\sum_{n=1}^{\infty} a_n$ did converge, then definitionally $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = S$.

Notice that $a_n = S_n - S_{n-1}$. Then by the limit laws:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0.$$

In other words: For $\sum_{n=1}^{\infty} a_n$ to be convergent, it is necessary that we have $\lim_{n\to\infty} a_n = 0$.

So, for example, any sequence such as

$$\sum_{n=1}^{\infty} \cos(n), \quad \sum_{n=1}^{\infty} \arctan(n), \quad \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right), \text{ etc}$$

must be divergent, just by checking that $\lim_{n\to\infty} a_n \neq 0$!

⚠ Be careful ⚠

This test <u>doesn't</u> tell us if a series $\sum_{n=1}^{\infty} a_n$ with $\lim_{n\to\infty} a_n = 0$ converges! In fact, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ passes this "test," while still diverging to ∞. In effect, the terms $a_n = \frac{1}{n}$ go to 0 <u>too slowly</u>. To see this precisely, one can make the following comparisons:

$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$
$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$
$= 1 + \frac{1}{2} + \frac{1}{2}$

$S_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$
$\geq 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)$

$S_{16} \geq 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)$

and so on.

So again: the moral of this example is that determining the convergence/divergence of a series can be tricky business.

Let's do some examples

1) Converge or Diverge? $\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$

$a_n = 2^{2n}3^{1-n} = 3\cdot\left(\frac{4}{3}\right)^n \rightsquigarrow \sum a_n$ diverges since $|r| = \frac{4}{3} > 1$.

2) Suppose for $\sum_{n=1}^{\infty} a_n$, we know somehow that $S_N = \frac{n-1}{n+1}$. What is a_n? Does $\sum a_n$ converge?

$a_n = S_n - S_{n-1} = \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{2}{n(n+1)}$, and $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = 1$.

It's sometimes helpful to know that for certain types of series, $\lim_{n\to\infty} a_n = 0$ is enough to guarantee convergence. Precisely, we have:

Alternating Series Test: If $\sum_{n=1}^{\infty} a_n$ is a series whose terms alternate between non-negative and non-positive, with $|a_{n+1}| \le |a_n|$, then $\sum_{n=1}^{\infty} a_n$ converges iff $\lim_{n\to\infty} a_n = 0$.

So, for example, the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

does converge!

As another example: $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{n^3+1}$: Converge or Diverge?

Check: alternating? Yes. $|a_{n+1}| \le |a_n|$? Yes: If $f(x) = \frac{x^2}{x^3+1}$, $f'(x) < 0$ on $(\sqrt[3]{2}, \infty)$

$\lim_{n\to\infty} (-1)^{n+1}\frac{n^2}{n^3+1} = 0$? Yes:

Here's the moral of this test: If your series looks like $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ with $b_n \ge 0$, $b_{n+1} \ge b_n$, and $\lim_{n\to\infty} b_n = 0$, then it must be convergent!

In fact, it's easy to show that $\left|\sum_{n=1}^{\infty} (-1)^{n+1} b_n - S_N\right| \le b_{n+1} \to 0$ $\forall n$.

Lastly for today, let's cover a few theorems about the convergence of series:

First, here are some "series laws" that follow from the "limit laws" we met for sequences earlier in Lecture 16:

Series Laws: Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series:

1) $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
2) For any fixed $c \in \mathbb{R}$, $\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n$.

And, just as for sequences, we have **comparison tests** for series too:

Comparison Laws: Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are such that all $a_n, b_n \geq 0$ (at least after *some* time $N \geq 1$):

1) If $a_n \leq b_n$ for all $n \geq N$ and $\sum b_n$ is convergent, then so is $\sum a_n$.
2) If $a_n \geq b_n$ for all $n \geq N$ and $\sum a_n$ is divergent, then so is $\sum b_n$.

Examples:

1) $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)} + \frac{1}{2^n}\right) = 1 + 3 \cdot 1 = 4$, since
   - $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1/2}{1-1/2} = 1$ is a geometric series
   - $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(\frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N} - \frac{1}{N-1}\right)$
     ↖ partial fractions! "telescoping series"
     $= \lim_{N\to\infty}\left(1 + \frac{1}{N}\right) = 1$

2) Converge or diverge? $\sum_{n=1}^{\infty} \frac{q^n}{3+10^n}$

$0 \leq a_n = \frac{q^n}{3+10^n} \leq \frac{q^n}{10^n} = \left(\frac{q}{10}\right)^n =: b_n$.

Since $\sum_{n=1}^{\infty} b_n = \frac{9/10}{1-9/10} = 9$, $\sum_{n=1}^{\infty} a_n \leq 9$ converges!

Another test which is sometimes useful is the following:

Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series so that $a_n, b_n > 0$ (at least after some time $N \geq 1$). If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = C \qquad (C > 0, \text{ finite})$$

then either

- both series converge
- both series diverge

Proof: Choose $m < C < M$, and $N \geq 1$ s.t. $\forall n \geq N$, $m < \frac{a_n}{b_n} < M$
$\Rightarrow m\sum b_n < \sum a_n < M\sum b_n$.
Follows from this...

Exercise: (Challenge) Show the following:

(1) Suppose $c = 0$. If $\sum b_n$ is convergent, then so is $\sum a_n$.

(2) Suppose $c = \infty$. If $\sum b_n$ is divergent, then so is $\sum a_n$.

Example: (1) Converge or Diverge? $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$

Let $a_n = \frac{1}{2^n - 1}$ and $b_n = \frac{1}{2^n}$ (convergent). Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = 1$.

(2) Converge or Diverge? $\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$

Let $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$ and $b_n = \frac{2n^2}{\sqrt{n^5}} = \frac{2}{n^{1/2}}$ (leading order terms in numerator, denominator)

Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5+n^5}} = \lim_{n\to\infty} \frac{2 + 3/n}{2\sqrt{\frac{5}{n^5} + 1}} = 1$.

So: How about $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^{1/2}}$? (Diverges since $1/2 \leq 1$)

A prelude to Monday: When does a series like $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge ($p > 0$)?

(1) Graph the sequence

(2) Realize the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ as a sum of areas of rectangles

(3) Bound this area with an improper integral which converges/diverges.

